MATH 512 HOMEWORK 1

Due Monday, Feb 14.

Recall that $I \subset \mathcal{P}(\kappa)$ is an *ideal* if $A \subset B$ and $B \in I$ implies $A \in I$; and if A, B are in I, then so is $A \cap B$. Given an ideal I, the dual filter is $F = \{A \subset \kappa \mid \kappa \setminus A \in I\}$. For example, the club filter on κ is the dual filter to the nonstationary ideal on κ.

I is normal if for all $S \subset \kappa$, $S \notin I$, and $f : S \to \kappa$ with $f(\alpha) \in \alpha$ for all $\alpha \in S$, we have that there is $T \subset S$, $T \notin I$, such that f is constant on T.

Problem 1. *Show that I is normal if and only if the dual filter is closed under diagonal intersections.*

Problem 2. *Suppose that U is a κ-complete normal ultrafilter on κ. Show that U contains all clubs of κ, and that every set in U is stationary.*

Problem 3. *Suppose that κ is supercompact and GCH holds below κ. (I. e. $\tau < \kappa \to 2^\tau = \tau^+$.) Show that GCH holds everywhere i.e. for all cardinals λ, $2^\lambda = \lambda^+$.*

For the problems below, let $\kappa < \lambda$ be regular cardinals and $\mathcal{P}_\kappa(\lambda) = \{x \subset \lambda \mid |x| < \kappa\}$. A set $A \subset \mathcal{P}_\kappa(\lambda)$ is club if:

- (unbounded) for all $x \in \mathcal{P}_\kappa(\lambda)$, there is $y \in A$ with $x \subset y$;
- (closed) for all $\alpha < \kappa$ and $x_0 \subset x_1 \subset ...x_\xi \subset ...$, $\xi < \alpha$ of elements in A, $\bigcup_{\xi<\alpha} x_\xi \in A$.

Problem 4. *Let κ be a regular uncountable cardinal and $\kappa < \lambda$. Show that if $\langle A_\alpha \mid \alpha < \lambda\rangle$ are clubs in $\mathcal{P}_\kappa(\lambda)$, then so is $\triangle_\alpha A_\alpha = \{x \in \mathcal{P}_\kappa(\lambda) \mid x \in \bigcap_{\alpha \in x} A_\alpha\}$.*

Problem 5. *Let $\kappa < \lambda$ be uncountable cardinals.*

(1) *Show that $A = \{x \in \mathcal{P}_\kappa(\lambda) \mid \kappa \cap x \in \kappa\}$ is club in $\mathcal{P}_\kappa(\lambda)$. For every $x \in A$, denote $\kappa_x := x \cap \kappa$.*
(2) *Suppose that C is a club in $\mathcal{P}_\kappa(\lambda)$, such that $C \subset A$, where A is the set above. Show that $\{\kappa_x \mid x \in C\}$ is club in κ.*

Recall that $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is a $\square_\kappa$ sequence iff:

(1) each C_α is a club subset of α,
(2) for each α, if $\mathrm{cf}(\alpha) < \kappa$, then $o.t.(C_\alpha) < \kappa$,
(3) for each α, if $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.

Problem 6. *Suppose that $\langle C_\alpha \mid \alpha \in Lim(\kappa^+)\rangle$ is a $\square_\kappa$ sequence. Show that there is no club $C \subset \kappa$ such that for all α, $C \cap \alpha = C_\alpha$.*

Hint: look at the order type of initial segments of such a C.

Problem 7. *Suppose that* $\langle C_\alpha \mid \alpha \in Lim(\kappa^+)\rangle$ *is a* $\Box_\kappa$ *sequence. Show that reflection at* κ^+ *fails.*

Hint: look at the function $\alpha \mapsto o.t.(C_\alpha)$.